INTERACTION BETWEEN A PERIODIC SYSTEM OF ELASTIC INCLUSIONS AND RECTILINEAR CRACKS IN AN ISOTROPIC MEDIUM

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The influence of a single inclusion on the stress state near an isolated crack has been investigated in a number of papers, to which there are references in [1].

According to fracture investigations, the inhomogeneity of real materials, which is practically inevitable during metallurgical and technological treatment results in the formation of a large quantity of defects (cracks, inclusions, pores), which are used in further construction and are the focus of rupture. Hence, it is of considerable interest to investigate the mutual influence of cracks oriented chaotically or specifically and inclusions.

The plane problem of elasticity theory is considered for an isotropic plane with circular holes filled with elastic washers from a foreign material soldered along the outline and weakened by rectilinear slots. The solution of this problem permits an estimate of the influence of the mutual disposition of a system of cracks and inclusions on the criteron (stress intensity coefficient) of the beginning of crack growth. This problem is also of interest for the theory of rupture of composite materials.

§1. Let there be a plane with circular holes of radius λ ($\lambda < 1$) and centers at the points

$$P_m = m\omega \ (m = 0, \pm 1, \pm 2, \ldots), \ \omega = 2.$$

The circular holes are filled with washers of a foreign elastic material soldered along the outline. The isotropic plane is weakened by a periodic system of rectilinear slits, as shown in Fig. 1. The lips of the slits are free of external forces. The mean stresses $\sigma_x = \sigma_x^\infty$, $\sigma_y = \sigma_y^\infty$, $\tau_{xy} = 0$ (tension at infinity) hold in the plane.

By virtue of the symmetry of the boundary conditions and the geometry of the domain occupied by the medium, the stresses are periodic functions with the period ω.

To solve the problem, a method developed for the solution of a periodic elastic problem, and a method [2, 3] for constructing explicit Kolosov–Muskhelishvili potentials corresponding to unknown normal displacements along the slits are combined in a natural way.

Let $N - iT$ denote a self-equilibrated, symmetric system of forces relative to the coordinate axes, which act on the washer from the plane. Considering $N - iT$ to be expanded in a Fourier series on the washer outline $|\tau| = \lambda$, we obtain

$$N - iT = \sum_{k=-\infty}^{\infty} A_{2k} e^{2ki\theta}, \ \mathrm{Im}\, A_{2k} = 0. \tag{1.1}$$

The functions $\Phi_0(z)$, $\Psi_0(z)$, describing the stress–strain state of the inclusion, are analytic in the inner circle $|\tau| = \lambda$ and can be represented by the series [4]

$$\Phi_0(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}, \quad \Psi_0(z) = \sum_{k=0}^{\infty} c_{2k} z^{2k}, \tag{1.2}$$

where

$$a_0 = \frac{A_0}{2}, \quad a_{2k} = \frac{A_{-2k}}{\lambda^{2k}} \quad (k = 1, 2, \ldots);$$

$$c_{2k} = -(2k+1)\frac{A_{-2k-2}}{\lambda^{2k}} - \frac{A_{2k+2}}{\lambda^{2k}} \quad (k = 0, 1, \ldots).$$

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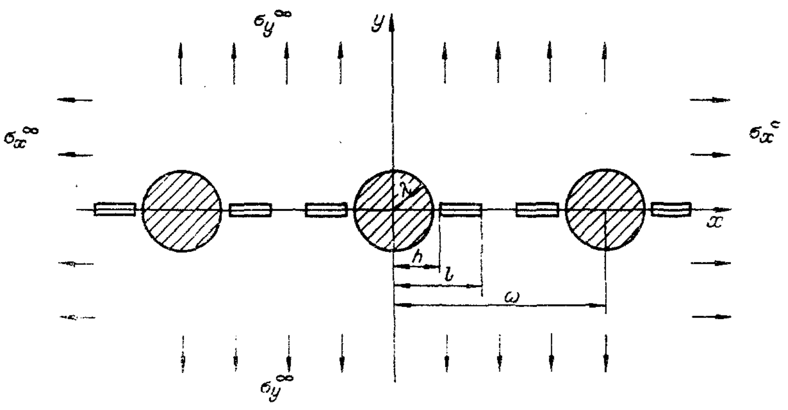

Fig. 1

The potentials $\Phi_0(z)$, $\Psi_0(z)$ permit finding the relationship

$$-2\mu_1 i e^{i\theta}\frac{d}{ds}(u_0 - iv_0) = A_0\frac{1-\varkappa_1}{\text{[illegible]}} + \sum_{k=1}^{\infty} A_{2k}e^{2ki\theta} - \sum_{k=1}^{\infty}\varkappa_1 A_{-2k}e^{-2ki\theta}, \tag{1.3}$$

where u_0, v_0 are the appropriate displacements of points of the inclusion contour, and μ_1, $\varkappa_1$ are coefficients characterizing the material of the inclusions. To determine the still unknown quantities A_{2k} ($k = 0, \pm 1, \ldots$), let us examine the solution for a plane. We see the complex potentials $\Phi(z)$, $\Psi(z)$ in the plane in the form

$$\Phi(z) = \Phi_1(z) + \Phi_2(z), \quad \Psi(z) = \Psi_1(z) + \Psi_2(z); \tag{1.4}$$

$$\Phi_1(z) = \frac{1}{2\omega}\int_L g(t)\operatorname{ctg}\frac{\pi}{\omega}(t-z)\,dt;$$

$$\Psi_1(z) = -\frac{\pi z}{2\omega^2}\int_L g(t)\sin^{-2}\frac{\pi}{\omega}(t-z)\,dt; \tag{1.5}$$

$$\Phi_2(z) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + \alpha_0 + \sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!},$$

$$\Psi_2(z) = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty) + \sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}s^{(2k+1)}(z)}{(2k+1)!}. \tag{1.6}$$

The integrals in (1.5) are taken over the line $L = \{[-l, -h] + [h, l]\}$,

$$g(x) = \frac{2\mu}{\varkappa+1}\frac{d}{dx}[h(x)],$$

$h(x) = v(x, +0) - v(x, -0)$ on L [by virtue of symmetry $h(x) = h(-x)$], $g(x)$ is the desired function, $\rho(z)$ is a periodic function, and $S(z)$ is a special meromorphic function [5].

An additional condition, resulting from the physical meaning of the problem

$$\int_{-l}^{-h} g(t)\,dt = 0, \quad \int_h^l g(t)\,dt = 0 \tag{1.7}$$

should be added to the relationships (1.4)-(1.6).

Now, let us present the dependence which the coefficients of (1.6) should satisfy. There follows from the symmetry conditions for the stress state in a plane relative to the coordinate axes with the Kolosov-Muskhelishvili formulas taken into account

$$\Phi(\bar z) = \overline{\Phi(z)},\ \Phi(-z) = \Phi(z),\ \Psi(\bar z) = \overline{\Psi(z)},\ \Psi(-z) = \Psi(z).$$

We hence find

$$\operatorname{Im}\alpha_{2k} = \operatorname{Im}\beta_{2k}, \quad k = 1, 2, \ldots \tag{1.8}$$

It can be seen that (1.4)-(1.6) determine the class of symmetric problems with a periodic stress distribution.

From the condition that the principal vector of all the forces acting on the arc connecting two congruent points is D is constant there follows

$$\alpha_0 = \frac{\pi^2}{24}\beta_2\lambda^2. \tag{1.9}$$

The unknown function g(x) and the constants α_{2k}, β_{2k} should be determined from the boundary conditions

$$\Phi(\tau) + \overline{\Phi(\tau)} - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,e^{2i\theta} = \sum_{k=-\infty}^{\infty} A_{2k}e^{2ki\theta}; \tag{1.10}$$

$$\Phi(t) + \overline{\Phi(t)} + t\overline{\Phi'(t)} + \overline{\Psi(t)} = 0, \tag{1.11}$$

where $\tau = \lambda_e i + m\omega$, m = 0, 1, 2, ...; and t is the affix of points of the edges of the slits.

To form the equations in the coefficients α_{2k}, β_{2k} of the functions $\Phi_2(z)$, $\Psi_2(z)$ we represent the boundary condition (1.10) in the form

$$\Phi_2(\tau) + \overline{\Phi_2(\tau)} - \left[\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)\right]e^{2i\theta} = f_1(\theta) + if_2(\theta) + \sum_{k=-\infty}^{\infty} A_{2k}e^{2ki\theta}, \tag{1.12}$$

where

$$f_1(\theta) + if_2(\theta) = -\Phi_1(\tau) - \overline{\Phi_1(\tau)} + \left[\bar{\tau}\Phi_1'(\tau) + \Psi_1(\tau)\right]e^{2i\theta}. \tag{1.13}$$

Relative to the function $f_1(\theta) + if_2(\theta)$ we consider it to be expanded in a Fourier series in $|\tau| = \lambda$. By virtue of symmetry, this series has the form

$$\begin{gathered} f_1(\theta) + if_2(\theta) = \sum_{k=-\infty}^{\infty} B_{2k}e^{2ki\theta}, \quad \operatorname{Im} B_{2k} = 0, \\ B_{2k} = \frac{1}{2\pi}\int_0^{2\pi}(f_1 + if_2)\,e^{-2ki\theta}d\theta \quad (k = 0, \pm 1, \pm 2, \ldots). \end{gathered} \tag{1.4}$$

Substituting (1.13) here and interchanging the order of the integration, we find after having evaluated the integrals by using residues

$$B_{2k} = -\frac{1}{2\omega}\int_L g(t)\,f_{2k}(t)\,dt. \tag{1.15}$$

The functions $f_{2k}(t)$ are determined for $\varepsilon = 1$ by the relationships

$$\begin{gathered} f_0(t) = (1+\varepsilon)\,\gamma(t), \quad f_2(t) = -\frac{\lambda^2}{2}\gamma^{(2)}(t), \\ f_{2k}(t) = -\frac{\lambda^{2k}(2k-1)}{(2k)!}\gamma^{(2k)}(t) + \frac{\lambda^{2k-2}}{(2k-3)!}\gamma^{(2k-2)}(t) \quad (k = 2, 3, \ldots), \\ f_{-2k}(t) = \frac{\varepsilon\lambda^{2k}}{(2k)!}\gamma^{(2k)}(t) \quad (k = 1, 2, \ldots), \\ \gamma(t) = \operatorname{ctg}\frac{\pi}{\omega}t. \end{gathered} \tag{1.16}$$

Substituting their expansions in Laurent series in the neighborhood of z = 0 [5] instead of $\Phi_2(\tau)$, $\Phi_2(\tau)$, $\Phi_2'(\tau)$, and $\psi_2(\tau)$ in the left side of the boundary condition (1.12), and the Fourier series (1.14) instead of $f_1 + if_2$ in the right side of (1.12), and equating coefficients of identical powers of $e^{i\theta}$, we obtain two infinite systems of linear algebraic equations in the coefficients α_{2k}, β_{2k}. After certain manipulations, we arrive [6] at an infinite system of linear algebraic equations in α_{2k}

$$\alpha_{2j+2} = \sum_{k=0} a_{j,k}\alpha_{2k+2} + b_j \qquad (j = 0, 1, 2, \ldots), \tag{1.17}$$

where $\varepsilon = 1$ in (1.17).

Because of their awkwardness, the coefficients $a_{j,k}$, b_j are not presented. Taking (1.3) into account, we write the condition of equality of the derivatives of the displacement in the plane and the washer on the contour $|\tau| = \lambda$ with respect to the arc

$$\begin{gathered} -\varkappa\overline{\Phi(\tau)} + \Phi(\tau) - [\bar{\tau}\Phi'(\tau) + \Psi(\tau)]\,e^{2i\theta} = f(\theta), \\ f(\theta) = \frac{\mu}{\mu_1}\left[A_0\frac{1-\varkappa_1}{2} + \sum_{k=1}^{\infty} A_{2k}e^{2ki\theta} - \varkappa_1\sum_{k=1}^{\infty} A_{-2k}e^{-2ki\theta}\right]. \end{gathered} \tag{1.18}$$

Let us represent the boundary condition (1.18) in the form

$$-\varkappa\overline{\Phi_2(\tau)} + \Phi_2(\tau) - [\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)]\,e^{2i\theta} = f_1^*(\theta) + if_2^*(\theta) + f(\theta); \tag{1.19}$$

$$f_1^*(\theta) + if_2^*(\theta) = \varkappa\overline{\Phi_1(\tau)} - \Phi_1(\tau) + [\bar{\tau}\Phi_1'(\tau) + \Psi_1(\tau)]\,e^{2i\theta}. \tag{1.20}$$

Proceeding with the function (1.20) and the boundary condition (1.19) in exactly the same manner as had been done with (1.13) and (1.12), we obtain the system (1.17) to determine the coefficients α_{2k+2} for $\varepsilon = -\varkappa$, in the right side of this system

$$A_0' = A_0^* = (\varkappa - 1)\frac{\sigma_x^\infty + \sigma_y^\infty}{4} - A_0\frac{(\varkappa_1 - 1)\mu}{2\mu_1} + B_0',$$

$$A_2' = A_2^* = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \frac{\mu}{\mu_1}A_2 + B_2',$$

$$A_{2k}' = A_{2k}^* = \frac{\mu}{\mu_1}A_{2k} + B_{2k}', \quad A_{-2k}' = A_{-2k}^* = -\varkappa_1\frac{\mu}{\mu_1}A_{-2k} + B_{-2k}'.$$

The quantities B_{2k}' are determined by (1.15), and the functions $f_{2k}(t)$ in this formula are found from (1.16) for $\varepsilon = -\varkappa$.

Using the method in [7], we obtain an infinite system of linear algebraic equations to determine the constants A_{2k}

$$A_{2j+2} = \sum_{k=0}^{\infty} d_{j,k}A_{2k+2} + T_j \qquad (j = 0, 1, \ldots), \tag{1.21}$$

as well as relationships permitting the coefficients α_{2k}, A_{-2k}, A_0 to be found in terms of the A_{2k}. Finally, by using the coefficients listed, the coefficients β_{2k} are found in terms of the A_{2k}. The quantities $d_{j,k}$ and T_j in the system (1.21) are not presented because of their awkwardness. Requiring that the functions (1.4) satisfy the boundary conditions on the edge of the slit L, we obtain a singular integral equation in g(x)

$$\begin{gathered} \frac{1}{\omega}\int_L g(t)\,\mathrm{ctg}\,\frac{\pi}{\omega}(t - x)\,dt + H(x) = 0, \\ H(x) = 2\Phi_2(x) + x\Phi_2'(x) + \Psi_2(x). \end{gathered} \tag{1.22}$$

The relationships connecting α_{2k} and β_{2k} in terms of A_{2k}, and the system (1.21) in combination with the singular equation (1.22) are the main equations of the problem which permit determination of the function g(x) and the coefficients α_{2k+2}, β_{2k+2}.

Let us recall that the function H(x) as well as the systems (1.17) and (1.21) contain the coefficients B_{2k} and B_{2k}' which depend on the desired function g(t). The system (1.21) and equation (1.22) turn out to be related and should be solved jointly.

By knowing the functions g(x), $\Phi_2(z)$, and $\Psi_2(z)$, the stress–strain state of the plate can be found. By changing the stiffness ratio between the inclusion and the plane, all the versions can be obtained, starting with a force-free circular hole and ending with an absolutely stiff inclusion.

We will have the formula

$$K_I = \pm \lim_{x \to c} \left[\sqrt{2\pi |x-c|}\, g(x) \right],$$

for the stress intensity coefficient K_I [8] at the crack apex, where the upper sign is taken for c = h and the lower for c = l.

For h = λ the behavior of the normal stresses depends on the kind of boundary conditions given along the outline of the circular holes. Two fundamental cases should here be distinguished.

1. The hole is filled by an elastic core; i.e., the crack emerges on the boundary of the inclusion. In this case, the singularity at the terminus (x = h) depends [8, 9] on the Poisson ratio ν_1 and the shear modulus μ_1 of the material of the inclusion.

2. The hole is not at all filled. In the case under consideration, the crack emerges on the surface of a hole free of external forces at one end x = h. In this case the stresses at the terminus x = h are bounded and have a singularity at the other end.

If an expansion of the function $\operatorname{ctg} \frac{\pi}{\omega} z$ is used by taking into account that g(x) = −g(−x) and by using a change of variable, then (1.22) can be given the standard form

$$\frac{1}{\pi} \int_{-1}^{1} \frac{p(\tau)\, d\tau}{\tau - \eta} + \frac{1}{\pi} \int_{-1}^{1} p(\tau) B(\eta, \tau)\, d\tau + H_*(\eta) = 0, \qquad (1.23)$$

where

$$p(\tau) = g(t); \quad H_*(\eta) = H(x); \quad B(\eta, \tau) = -d \sum_{j=0}^{\infty} g_{j+1} \left(\frac{l}{2}\right)^{2j+2} u_0^j A_j;$$

$$d = \frac{1}{2}(1 - \lambda_1^2); \qquad u = d(\tau + 1) + \lambda_1^2;\ u_0 = d(\eta+1) + \lambda_1^2; \quad \lambda_1 = \frac{h}{l};$$

$$A_j = \left[(2j+1) + \frac{(2j+1)(2j)(2j-1)}{1 \cdot 2 \cdot 3} \left(\frac{u}{u_0}\right) + \ldots + \left(\frac{u}{u_0}\right)^j \right].$$

A method developed in [10] is used to solve (1.23). We represent the solution in the form $p(\eta) = p_*(\eta)/\sqrt{1-\eta^2}$.

The function $p_*(\eta)$ is replaced by a Lagrange interpolation polynomial constructed by means of Chebyshev nodes. Using the quadrature formulas

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{p(\tau)\, d\tau}{\tau - \eta} = \frac{1}{n \sin\theta} \sum_{\nu=1}^{n} p_\nu^0 \sum_{m=0}^{n-1} \cos m\theta_\nu \sin m\theta,$$

$$\frac{1}{2\pi} \int_{-1}^{1} p(\tau) B(\eta, \tau)\, d\tau = \frac{1}{2n} \sum_{\nu=1}^{n} p_\nu^0 B(\eta, \tau_\nu), \quad p_\nu^0 = p_0(\eta_\nu), \quad \eta_m = \cos\theta_m,$$

$$B_{2k} = -\frac{d}{2n} \sum_{\nu=1}^{n} p_\nu^0 f_{2k}^*(\tau_\nu), \quad \tau_\nu = \eta_\nu, \quad \theta_m = \frac{2m-1}{2n}\pi, \quad m = 1, 2, \ldots, n,$$

$$\int_{-1}^{1} \frac{p(\tau)\, d\tau}{\sqrt{d(1+\tau) + \lambda_1^2}} = \frac{\pi}{n} \sum_{\nu=1}^{n} \frac{p_\nu^0}{\sqrt{d(1+\tau_\nu) + \lambda_1^2}}$$

permits replacement of the fundamental equations by an infinite system of linear algebraic equations in the approximate values p_ν^0 of the desired function at the nodal points, as well as in the coefficients A_{2k}. By using relationships connecting α_{2k} and β_{2k} in terms of A_{2k}, the constants α_{2k}, β_{2k} are hence eliminated from the expression $H_*(\eta)$.

After having found the values of p_ν^0, the stress intensity coefficient K_I is determined by the following relationships:

$$K_I^h = \sqrt{\frac{\pi l (1 - \lambda_1^2)}{\lambda_1}}\, \frac{1}{2n} \sum_{\nu=1}^{n} (-1)^{\nu+n} p_\nu^0 \operatorname{tg} \frac{\theta_\nu}{2},$$

$$K_I^l = \sqrt{\pi l (1-\lambda_1^2)} \frac{1}{2n} \sum_{\nu=1}^{n} (-1)^\nu p_\nu^0 \operatorname{ctg} \frac{\theta_\nu}{2}.$$

In the second case, when the crack emerges on the surface of a free hole, the solution of (1.23) is sought in the form

$$p(\eta) = \sqrt{\frac{1+\eta}{1-\eta}} p_0(\eta).$$

The quantities B_{2k} are given this time by the formulas

$$B_{2k} = -\frac{d}{2n} \sum_{\nu=1}^{n} p_\nu^0 (1+\tau_\nu) f_{2k}^*(\tau_\nu),$$

and the stress intensity coefficient is determined by the relationship

$$K_I^l = \sqrt{\pi l (1-\lambda_1^2)} \frac{1}{n} \sum_{\nu=1}^{n} (-1)^\nu p_\nu^0 \operatorname{ctg} \frac{\theta_\nu}{2}.$$

Computations were performed to realize the method elucidated numerically. The plate tension by the constant forces $\sigma_y^\infty (\sigma_x^\infty = 0)$ in a direction perpendicular to the slit lines was investigated. It was assumed that n = 20 and 30, which corresponds to partitioning the interval into 20 and 30 Chebyshev nodes, respectively. The system (1.21) was truncated to five equations. The systems mentioned were solved by the Gauss method. The solutions agree to the accuracy of the sixth place.

In the first case we have for the stress intensity coefficient

$$K_I^h = \sigma_y^\infty \sqrt{\pi l} \sqrt{\frac{(1-\lambda_1^2)}{\lambda_1}} F_1(\lambda, h, l),$$

$$K_I^l = \sigma_y^\infty \sqrt{\pi l} \sqrt{1-\lambda_1^2} F_2(\lambda, h, l).$$

Results of computing the functions $F_1(\lambda, h, l)$ and $F_2(\lambda, h, l)$ as the spacing h changes for the two limit cases of an absolutely stiff inclusion (values given in the numerator), and an absolutely flexible inclusion (holes not filled at all) are presented in Table 1. The crack length was assumed constant $l - h = 0.3$ in the computations. For any elastic inclusion, the picture of the stress state will be intermediate between these two limit cases. The investigation showed that taking account of the interaction between the system of cracks and inclusions increases the stress intensity coefficient significantly as compared with a single inclusion and an isolated crack.

In the second case, when the crack emerges at one end x = h on the surface of a hole free of external forces, we have for the stress intensity coefficient

$$K_I^l = \sigma_y^\infty \sqrt{\pi l} \sqrt{1-\lambda_1^2} F_3(\lambda, l).$$

TABLE 1

λ		h							
		0,21	0,25	0,29	0,33	0,37	0,41	0,45	0,49
0,2	$F_1(\lambda, h, l)$	$\frac{0,243}{3,169}$	$\frac{0,519}{1,528}$	$\frac{0,650}{1,249}$	$\frac{0,717}{1,153}$	$\frac{0,762}{1,101}$	$\frac{0,798}{1,069}$	$\frac{0,825}{1,051}$	$\frac{0,849}{1,042}$
	$F_2(\lambda, h, l)$	$\frac{0,963}{1,723}$	$\frac{1,045}{1,542}$	$\frac{1,074}{1,454}$	$\frac{1,089}{1,391}$	$\frac{1,097}{1,346}$	$\frac{1,104}{1,315}$	$\frac{1,112}{1,295}$	$\frac{1,123}{1,285}$
λ		h							
		0,31	0,35	0,39	0,43	0,47	0,51	0,55	0,59
0,3	$F_1(\lambda, h, l)$	$\frac{0,219}{6,948}$	$\frac{0,501}{2,553}$	$\frac{0,616}{1,618}$	$\frac{0,659}{1,421}$	$\frac{0,713}{1,329}$	$\frac{0,758}{1,274}$	$\frac{0,797}{1,241}$	$\frac{0,834}{1,225}$
	$F_2(\lambda, h, l)$	$\frac{0,894}{2,007}$	$\frac{0,921}{1,724}$	$\frac{0,962}{1,609}$	$\frac{0,997}{1,544}$	$\frac{1,024}{1,497}$	$\frac{1,049}{1,469}$	$\frac{1,078}{1,459}$	$\frac{1,117}{1,473}$

TABLE 2

l \ λ	0,1	0,2	0,3	0,4	0,5
0,15	2,828				
0,20	2,420				
0,25	2,274	3,351			
0,30	2,218	2,863			
0,35	2,203	2,622	3,666		
0,40	2,213	2,500	3,199		
0,45	2,244	2,445	2,929	3,929	
0,50	2,286	2,432	2,789	3,486	
0,55	2,346	2,452	2,717	3,237	4,233
0,60	2,427	2,501	2,699	3,095	3,803
0,65	2,531	2,581	2,728	3,032	3,586
0,70	2,667	2,696	2,804	3,039	3,479
0,75	2,849	2,860	2,935	3,116	3,469
0,80	3,099	3,094	3,140	3,276	3,563
0,85	3,459	3,344	3,457	3,555	3,789
0,90	4,011	3,969	3,963	4,027	4,221

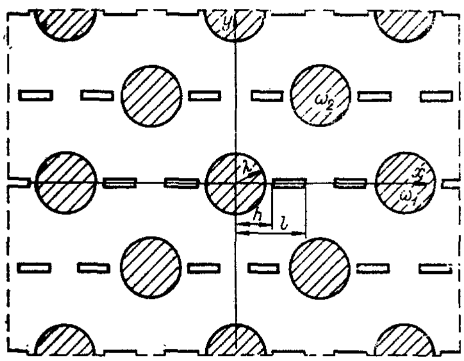

Fig. 2

Results of computations for the function $F_3(\lambda, l)$ are given in Table 2. In contrast to the case of two cracks issuing from a single hole, the interaction between a system of cracks and holes results in the growth of the intensity coefficient as the crack length increases.

2. Let there be a doubly periodic lattice with circular holes of radius $\lambda (\lambda < 1)$ and centers at the points

$$P_{mn} = m\omega_1 + n\omega_2 \ (m, \ n = 0, \ \pm 1, \pm 2, \ldots),$$
$$\omega_1 = 2, \ \omega_2 = 2le^{i\alpha}, \ l > 0, \ \operatorname{Im}\omega_2 > 0.$$

Circular holes of the lattice are filled by washers from a foreign elastic material soldered along the outline. The lattice is weakened by a doubly periodic system of rectilinear slits as shown in Fig. 2. The edges of the slits are free of external forces. The mean stresses $\sigma_x = \sigma_x^\infty$, $\sigma_y = \sigma_y^\infty$, $\tau_{xy} = 0$ (tension at infinity) hold in the lattice.

By virtue of symmetry of the boundary conditions and the geometry of the domain D occupied by the medium, the stresses are doubly periodic functions with the fundamental periods ω_1 and ω_2.

To solve the problem, a method [7] developed to solve the doubly periodic elastic problem is combined in a natural way with the method [2, 3, 11] of explicitly constructing the Kolosov–Muskhelishvili potentials corresponding to the unknown normal displacements along the slits.

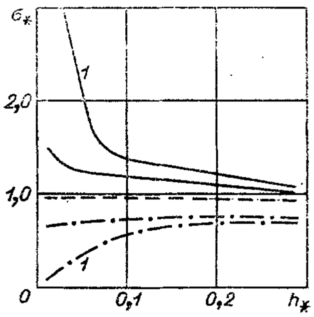

Fig. 3

We seek the complex potentials $\Phi(z)$ and $\Psi(z)$ in the lattice in the form (1.4), where

$$\Phi_1(z) = \frac{1}{2\pi}\int_L g(t)\,\zeta(t-z)\,dt + A,$$
$$\Psi_1(z) = \frac{1}{2\pi}\int_L g(t)\left[\zeta(t-z) + Q(t-z) - t\gamma(t-z)\right]dt + B; \tag{2.1}$$

$$\Phi_2(z) = \frac{1}{4}\left(\sigma_x^\infty + \sigma_y^\infty\right) + \sum_{k=0}^{\infty} \alpha_{2k+2}\frac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!},$$
$$\Psi_2(z) = \frac{1}{2}\left(\sigma_y^\infty - \sigma_x^\infty\right) + \sum_{k=0}^{\infty} \beta_{2k+2}\frac{\lambda^{2k+2}\gamma^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2}\frac{\lambda^{2k+2}Q^{(2k+1)}(z)}{(2k+1)!}. \tag{2.2}$$

Here $\gamma(z)$ and $\zeta(z)$ are Weierstrass functions; $Q(z)$ is a special meromorphic function [7]; and A and B are constants. In the case under consideration the relationships (1.1)-(1.3), (1.7), and (1.8) remain valid.

The condition of constancy of the principal vector of all forces acting on an arc connecting two congruent points in D, with (1.7) and properties of the functions $\gamma(z)$, $\zeta(z)$, and $Q(z)$ at the congruent points taken into account, will result in the relationship

$$(A + \overline{A})\omega_k + \overline{B}\overline{\omega}_k = \delta_k a + \overline{\gamma}_k a + \overline{\delta}_k(a + \overline{a}) + \alpha_2\lambda^2(\delta_k + \overline{\gamma}_k) + \beta_2\lambda^2\overline{\delta}_k$$
$$(k = 1,\ 2),$$
$$a = -\frac{1}{2\pi}\int_L tg(t)\,dt. \tag{2.3}$$

The notation for the constants of the doubly periodic lattice corresponds with that used in [7]. The constants A and B are determined from the system (2.3), where A and B are real.

It can be seen that the functions (1.4), (2.1), and (2.2) determine the class of symmetric problems with a doubly periodic stress distribution under the condition (1.8). The unknown $g(x)$ and the constants α_{2k} and β_{2k} should be determined from the boundary conditions (1.10), (1.11), where $\tau = \lambda e^{i\theta} + m\omega_1 + n\omega_2$, m, n = 0, ±1, ±2, ... To obtain the fundamental equations of the problem, the discussion in Sec. 1 should be repeated.

In this case

$$B_0 = -2A + B_0^*, \quad B_2 = B + B_2^*, \ B_{2k} = B_{2k}^* \ (k = -1, \pm 2, \pm \ldots),$$
$$B_{2k}^* = -\frac{1}{2\pi}\int_L g(t) f_{2k}(t)\,dt, \quad f_0(t) = (1+\varepsilon)\,\zeta(t),$$
$$f_2(t) = \frac{\lambda^2}{2}\gamma'(t) + t\gamma(t) - \zeta(t) - Q(t),$$

$$f_{2k}(t) = \frac{(2k-1)\lambda^{2k}}{(2k)!}\gamma^{(2k-1)}(t) + \frac{\lambda^{2k-2}}{(2k-2)!}[\gamma^{(2k-3)}(t) - $$
$$- Q^{(2k-2)}(t) + t\gamma^{(2k-2)}(t)] \quad (k = 2, 3, \ldots).$$

$$f_{-2k}(t) = -\frac{e\lambda^{2k}}{(2k)!}\gamma^{(2k-1)}(t) \quad (k = 1, 2, \ldots).$$

The singular integral equation has the form

$$\frac{1}{2\pi}\int_L g(t)K(t-x)\,dt + H(x) = 0 \text{ at } L, \tag{2.4}$$

where

$$K(x) = 3\zeta(x) + Q(x) - x\gamma(x);$$
$$H(x) = 2A + B + 2\Phi_2(x) + x\Phi_2'(x) + \Psi_2(x);$$
$$2A + B = \frac{1}{\omega_1}[(a + \alpha_2\lambda^2)(\delta_1 + \gamma_1) + (2a + \beta_2\lambda^2)\delta_1],$$

and the system in A_{2k} formally remains (1.21) as before. Using the expansions of $\gamma(z)$ and $\zeta(z)$ and $Q(z)$ in the fundamental period parallelogram [7], and also taking into account $g(x) = -g(-x)$ and using a change of variable, we reduce (2.4) to the form of (1.23).

Computations were performed to realize the method elucidated. The tension of a regular triangular lattice $\omega_1 = 2$, $\omega_2 = 2e^{(1/3)i\pi}$ by the constant forces $\sigma_y^\infty(\sigma_x^\infty = 0)$ in a direction perpendicular to the slit lines was investigated. Values of the limit (rupturing) forces were determined as a function of the geometric and physical parameters of the problem. Dependences of the critical load $\sigma_* = \sigma_y^\infty(\sqrt{\omega_1/2})/K_{Ic}$ on the distance $h_* = h - \lambda$ for both ends of the crack (curve 1 corresponds to the left end) are shown in Fig. 3 for $\lambda = 0.3$ on the basis of the results obtained in the case of a stiff inclusion with $\nu = 0.3$. Shown for comparison by dashes is the dependence of σ_* in the absence of inclusions (the inclusion and lattice materials are identical) for the same crack geometry, calculated by the method described, while the dependence in the case of an absolutely flexible inclusion (the holes not filled at all) is shown by a dash–dot line. The picture of the stress state will be intermediate between these two limit cases for any elastic inclusion. An investigation showed that the mutual influence of the system of cracks and inclusions increases the stress intensity coefficient considerably as compared with a single inclusion and an isolated crack. In the case of cracks emerging at one end at the free surface of a circular hole ($h = \lambda$), the stable development of a system of cracks (their mutual hardening) is observed for certain values of λ. It is curious to note that there is no possibility of stabilizing crack development for a doubly periodic system of cracks with the same geometry but without circular holes ($\lambda = 0$).

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